Submitted to SIAM J. Math. Anal.

(4)

An H_0^m interpolation result

S. Jensen[*†]

November 14, 1989

Abstract

This paper presents a proof of an interpolation result related to the approximation theory for higher order finite element or spectral methods when C^1 (or higher) regularity is convenient for the finite dimensional subspaces. This can be a natural choice for example for Stokes problem, the biharmonic problem or higher order plate- and shell models. We show that one gets the same intermediate spaces whether one 1) interpolates between two Sobolev spaces defined on a domain with nonsmooth boundary first and then enforces the homogeneous boundary conditions afterwards or 2) interpolates between two Sobolev spaces where the homogeneous boundary conditions are enforced throughout the interpolation proces.

Key words. Interpolation, Peetre, boundary conditions, nonsmooth domains, small angle elliptic regularity.

AMS(MOS) subject classifications. 65N30, 46E35, 35J40, 35B65.

1. Introduction

The aim of this note is to prove an interpolation result for domains in $\mathbf{R}^2$ with finitely many corners and otherwise smooth boundary. We consider a bounded open set Ω of $\mathbf{R}^2$, whose boundary is a curvilinear polygon of class C^∞ (see [6]). We denote each of the C^∞ curves which constitute the boundary by $\overline{\Gamma_j}$ for some j ranging from 1 to N. The curve $\overline{\Gamma_{j+1}}$ follows $\overline{\Gamma_j}$ according to the positive orientation, on each connected component of Γ. We denote by C_j the vertex which is the end point of $\overline{\Gamma_j}$ and by α_j the measure of the angle at C_j (toward the interior of Ω). By a corner we mean a vertex C_j with an angle α_j not in the set $\{0, \pi, 2\pi\}$. The result is an extension to H_0^m of the one in [3], [1] for H_0^1 which would be useful in approximation theory for Sobolev spaces, see [7], Remark 2.2.9, [14], Remark 4.2, [2], and [11], the line following (III.26) in the proof of Thm. III.2. For example, consider solving Stokes problem via

[*]Dept. of Math. and Stat.,UMBC, Baltimore, MD 21228

[†]Supported in part by Office of Naval Research under contract N00014-87-K-0427

the p version of the finite element method or a polynomial spectral method. Then the discrete velocity $\vec{U}_p$ is an elliptic projection onto a finite dimensional subspace Z_p of $Z = [H_0^1(\Omega)]^2 \cap \mathrm{Ker}(\mathrm{div})$ centering interest on the approximation problem. Introducing stream functions ($\vec{U} = \mathrm{rot}\phi, \vec{U}_p = \mathrm{rot}\phi_p$) will translate this approximation problem to $H_0^2(\Omega)$. Now one gets for free an energy estimate – $\|\phi - \phi_p\|_2$ bounded when $\phi \in H_0^2$ only – and there exists constructive approximation estimates – $\|\phi - \phi_p\|_2 \leq Cp^{2-t}\|\phi\|_t$ when $\phi \in H^t(\Omega) \cap H_0^2(\Omega)$ for $t > 7/2$, see [14]. Now, one wishes to interpolate between these spaces and hopes to get spaces that coincide in some sense with the ones predicted by regularity theory, but the trace constraints on a nonsmooth boundary makes this identification nontrivial. In general such an identification is useful for higher order finite element or spectral methods when C^1 (or higher) regularity is convenient for the finite dimensional subspaces. This can be a natural choice for example also for the biharmonic problem or higher order plate- and shell models.

Let $H^s(\Omega)$ be the standard Sobolev space of order s based on L_2 with corresponding norm $\|\cdot\|_s$. $H_0^m(\Omega)$ is the set of functions in $H^m(\Omega)$ for which the traces of the function and its normal derivatives up to order $m-1$ vanish on $\partial\Omega$.

We shall use the interpolation spaces of Peetre (see e.g. [4]) in the cases $1 \leq q \leq \infty$ where we define $[H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q}$ explicitely: For $u \in H^t \cap H_0^m$, we set

$$K(u,s) = \inf_{\substack{u = v + w \\ v \in H_0^m,\ w \in H^t \cap H_0^m}} (\|v\|_m + s\|w\|_t)$$

and we define the norm

$$\|u\|_{[\cdot,\cdot]_{\theta,q}} = \|s^{-1/q-\theta} K(u,s)\|_{L_q(0,\infty)}$$

Then

$$[H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q} = \{u \in H_0^m(\Omega) : \|u\|_{[\cdot,\cdot]_{\theta,q}} < \infty\}$$

$[H^t(\Omega), H^m(\Omega)]_{\theta,q}$ is defined similarly. Note that this space will be a Sobolev space if we choose $q = 2$ and in general a Besov space.

2. The interpolation result

We state and prove:

Proposition 1 *Let $\Omega \subseteq \mathbf{R}^2$ be piecewise C^∞ with finitely many corners of angles in $(0, 2\pi)\setminus\{\pi\}$. Then the following identity holds for all $\theta \in (0,1)$, $1 \leq q \leq \infty$ and $t \geq m$, $t \notin m + \{\frac{1}{2}, \ldots, m - \frac{1}{2}\}$, $m \in \mathbf{Z}_+$:*

$$[H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q} = [H^t(\Omega), H^m(\Omega)]_{\theta,q} \cap H_0^m(\Omega)$$

Proof: We follow the main ideas of [3] but have weights be unity for simplicity, see also [15] and [1].

The inclusion from left to right follows directly from the definition.

The proof of the reverse inclusion can through a partition of unity be reduced to considering a domain Ω with one corner of angle $\alpha \in (0, 2\pi) \setminus \{\pi\}$. We shall distinguish between two cases: whether $\alpha \in (0, \pi)$ or $(\pi, 2\pi)$.

Case 1: $\alpha \in (0, \pi)$. Then there exists a linear transformation from Ω to $\tilde{\Omega}$ with a corner of angle $\tilde{\alpha} < \min\{\omega_0, \pi\}$ where ω_0 will be introduced in the next section as a sufficiently small angle that a certain shift theorem will hold. Let L be the associated map of functions defined on $\tilde{\Omega}$ to functions defined on Ω. If $w \in H^t(\tilde{\Omega})$, then we let $\tilde{v} = P_{\tilde{\Omega}} w$ denote the solution to

$$\begin{gathered} (-\triangle)^m \tilde{v} + \tilde{v} = (-\triangle)^m w + w \text{ in } \tilde{\Omega} \\ \tilde{v} \in H_0^m(\tilde{\Omega}) \end{gathered} \tag{2.1}$$

Thus $P_{\tilde{\Omega}}$ is a projection from $H^m(\tilde{\Omega})$ to $H_0^m(\tilde{\Omega})$. As proven in the next section on regularity, there exists ω_0, dependent on m and t, such that the following shift theorem holds provided $\tilde{\alpha} < \omega_0$:

$$\|\tilde{v}\|_{H^t} \leq C \|(-\triangle)^m w + w\|_{H^{t-2m}}$$

In particular, $P_{\Omega} = L \circ P_{\tilde{\Omega}} \circ L^{-1} \in \mathcal{B}(H^m(\Omega), H_0^m(\Omega))$ and $P_{\Omega} \in \mathcal{B}(H^t(\Omega), H^t(\Omega) \cap H_0^m(\Omega))$. Thus, by interpolation,

$$P_{\Omega} \in \mathcal{B}([H^t(\Omega), H^m(\Omega)]_{\theta,q}, [H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q})$$

Since $P_{\Omega}|_{H_0^m(\Omega)} = I$ (the identity),

$$[H^t(\Omega), H^m(\Omega)]_{\theta,q} \cap H_0^m(\Omega) \subseteq [H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q} \tag{2.2}$$

Case 2: $\alpha \in (\pi, 2\pi)$. Let B be a ball centered at the corner and containing Ω. By E_{Ω}, we denote the Stein extension ([13] Chapter VI, Sec. 3) of functions on Ω to functions on B vanishing at ∂B. Let $\Omega^c = B \setminus \Omega$ and let E_{Ω^c} be the Stein extension of functions on Ω^c to all of B. Theorem 5 in [13] states that $E_{\Omega} \in \mathcal{B}(H^k(\Omega), H^k(B))$ and $E_{\Omega^c} \in \mathcal{B}(H^k(\Omega^c), H^k(B))$, $\forall k \in \mathbf{N}$. Now define

$$P_{\Omega} = E_{\Omega^c} \circ P_{\Omega^c} \circ E_{\Omega} + (I - E_{\Omega^c} \circ E_{\Omega})$$

with P_{Ω^c} being the the same operator as in Case 1. Then $P_{\Omega}|_{H_0^m(\Omega)} = I$ and

$$P_{\Omega} \in \mathcal{B}([H^t(\Omega), H^m(\Omega)]_{\theta,q}, [H^t(\Omega) \cap H_0^m(\Omega), H_0^m(\Omega)]_{\theta,q})$$

which ends the proof of the proposition.

////

Remark 1: We have explicitely excluded vertices of angles $0, \pi$, or 2π. In these cases it is not possible to map linearly onto a domain of sufficiently small angle. In case Ω were a polygon the exclusion only amounts to 2π.

Remark 2: The theorem and proof hold for conical points in $\mathbf{R}^3$ almost verbatim.

3. Regularity for small angles

In [10] it is stated that, given $k \in \mathbf{N}$, if the domain contains only corners of sufficiently small angles and $f \in H_0^k$, then the solution (u) of a Dirichlet problem with zero boundary conditions and a $2m$ order, elliptic operator ($Lu = f, u \in H_0^m$) belongs to H^{k+2m}. We present a proof here following and expanding upon the ideas in [10] pp 292-294 and [5] extending to the case where $f \in H^k$, $k \geq -m$. We use the notation of [10].

Let L be elliptic of order $2m$ and $u \in H_0^m$ be the solution of

$$Lu = f$$

In [10] a technique is used that involves a combination of 1) looking at $L_0(0, \frac{\partial}{\partial x})$: the principal part of the operator $L(x, \frac{\partial}{\partial x})$ with coefficients fixed at the origin, 2) changing to polar variables (r, ω), so that $L_0 u = f$ takes the form:

$$\sum_{i_1, i_2 = 0}^{2m} \frac{a_{i_1 i_2}(\omega)}{r^{i_1}} \frac{\partial^{2m-i_2} u}{\partial r^{2m-i_1} \partial \omega^{i_1 - i_2}} = f$$

3) making the change of the radial variable ($\varrho = \ln \frac{1}{r}$) so that $L_0 u = f$ now takes the form

$$\sum_{k_1 + k_2 = 1}^{2m} a_{k_1 k_2}(\omega) \frac{\partial^{k_1 + k_2} u}{\partial \varrho^{k_1} \partial \omega^{k_2}} = f \cdot e^{-2m\varrho} = F$$

and 4) taking the Fourier transform with respect to the "radial" variable (ϱ). The domain then consists of angles $\omega \in \widetilde{D}$ – an interval (for $\mathbf{R}^2$ – in higher dimensions a cylinder with smooth boundary). The final form of $L_0 u = f$ is

$$L_0(\omega, i\lambda, -\frac{\partial}{\partial \omega})\hat{u} = \hat{F}$$

The boundary conditions undergo similar transformations. Let $R(\lambda)$ be the resolvent operator – a meromorphic function of λ – associated with the resulting boundary value problem. In [10] and [9] it is shown that if $f \in H_0^k$, then $u \in H^m$ has the expansion

$$u = \sum_{h_0 < \mathrm{Im}\lambda_j < h} \sum_{s=0}^{n_j} \alpha_{js} r^{-i\lambda_j} \log^s r \sum_{q=0}^{s} P_{jsq}(r \log^q r) \tag{3.1}$$

$$+ \sum_{l=0}^{h_0} \sum_{p=0}^{\|\vec{n}\|_\infty} c_{lp} r^l \log^p r \, \psi_{lp}(\omega) + w \tag{3.2}$$

where $h_0 = -1 + m$, $h = -1 + k + 2m$, λ_j are the poles of $R(\lambda)$ of multiplicity n_j, P_{jsq} are polynomials of degree $[h - \mathrm{Im}\lambda_j]$ whose coefficients are C^∞ functions of ω as are ψ_{lp}, and $w \in H^{k+2m}$. From this expansion we see that the smoothness of u depends on the poles λ_j of the function $R(\lambda)$ which lie above the straight line $\mathrm{Im}\lambda = -1 + m$. We will show that,

Lemma 1 *Given any positive h, there exists ω_0 such that, if the angle of the corner is smaller than ω_0, then the strip $-1+m < Im\lambda < h$ contains no poles of $R(\lambda)$.*

Proof: Let h be given and λ_0 be a pole of $R(\lambda)$ lying in the strip $-1+m < \mathrm{Im}\lambda < h$. When $\lambda = \lambda_0$, there exists a nonzero solution $u_0(\omega)$ to

$$\widetilde{L_1}(\lambda, \omega, \frac{\partial}{\partial\omega})u_0 = 0 \quad \text{in } \widetilde{D}$$

$$u_0 = \frac{\partial u_0}{\partial\omega} = \cdots = \frac{\partial^{m-1}u_0}{\partial\omega^{m-1}} = 0 \quad \text{on } \tilde{\Gamma}$$

Now, $\widetilde{L_1}u_0 = L_0u_0 + \lambda L_1 u_0$, where the operator L_1 contains derivatives of order less than $2m$. Since this system is elliptic for all real λ, it is elliptic for $\lambda = 0$. So L_0 is elliptic. Let

$$I(u) = \int_{\widetilde{D}} (\widetilde{L_1}u)\bar{u}\,d\omega$$

which at u_0 is zero: integrate by parts

$$\begin{aligned}
&I_1(u_0) + I_2(u_0) + I_3(u_0) = \\
&\quad \int_{\widetilde{D}} \{[a_{mm}(\omega)\frac{\partial^m u_0}{\partial\omega^m}\frac{\partial^m \overline{u_0}}{\partial\omega^m} \\
&\quad + \sum_{0<i+j<2m} \lambda^{i+j} a_{ij}(\omega)\frac{\partial^{m-i}u_0}{\partial\omega^{m-i}}\frac{\partial^{m-j}\overline{u_0}}{\partial\omega^{m-j}}] \\
&\quad + \lambda^{2m}a_{00}(\omega)u_0\overline{u_0}\}d\omega \\
&\quad = 0
\end{aligned}$$

By ellipticity of L_0,

$$|\mathrm{Re}I_1(u_0)| \geq \alpha_0 \int_{\widetilde{D}} |\frac{\partial^m u_0}{\partial\omega^m}|^2 d\omega - C_0 \int_{\widetilde{D}} |u_0|^2 d\omega$$

where α_0 and C_0 do not depend on u_0 or σ – the diameter of $\widetilde{D}$.

$$|I_2(u_0)| \leq \epsilon \int_{\widetilde{D}} |\frac{\partial^m u_0}{\partial\omega^m}|^2 d\omega + C(\epsilon)|\lambda|^{2(m-1)} \int_{\widetilde{D}} |u_0|^2 d\omega$$

as sketched in [10] and

$$|I_3(u_0)| \leq C|\lambda|^{2m} \int_{\widetilde{D}} |u_0|^2 d\omega$$

where C does not depend on σ. Also

$$\int_{\widetilde{D}} |\frac{\partial^m u_0}{\partial\omega^m}|^2 d\omega \geq \frac{C}{\sigma^{2m}} \int_{\widetilde{D}} |u_0|^2 d\omega$$

Upon contracting like terms, substituting this last inequality, and cancelling $\int_{\tilde{D}} |u_0|^2 d\omega$, we get from $|\mathrm{Re} I_1| \le |I_2| + |I_3|$

$$\alpha_1 \sigma^{-2m} \le 1 + |\lambda|^{2(m-1)} + |\lambda|^{2m}$$

for some $\alpha_1 > 0$ independent of σ. Since $\mathrm{Im}\lambda \in (-1+m, h)$, if $\lambda = \rho e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, given any $\epsilon > 0$, there exists σ sufficiently small so that

$$\theta \in (-\frac{\epsilon}{2m}, \frac{\epsilon}{2m}) \cup (\pi - \frac{\epsilon}{2m}, \pi + \frac{\epsilon}{2m})$$

and thus $2m\theta \in (-\epsilon, \epsilon) \cup (2m\pi - \epsilon, 2m\pi + \epsilon) = (-\epsilon, \epsilon)$ such that $\mathrm{Re}(\lambda^{2m}) > 0$. Having

$$|\mathrm{Re} I_3| \ge \frac{1}{2}|\lambda|^{2m} \int_{\tilde{D}} |u_0|^2 d\omega - C(m)|\mathrm{Im}\lambda|^{2m} \int_{\tilde{D}} |u_0|^2 d\omega$$

If we again contract, substitute, and cancel as before, but now in $|\mathrm{Re} I_1 + \mathrm{Re} I_3| \le |I_2|$ (using a_{00} and a_{mm} have same signs and $\mathrm{Re}(\lambda^{2m}) > 0$), we get

$$\sigma^{-2m} + C_1|\lambda|^{2m} \le C_2|\lambda|^{2(m-1)} + C_3$$

admitting no solutions λ for sufficiently small σ.

////

Remark 3: Another way of proving this lemma for the biharmonic operator is by checking that one can choose the angle $\tilde{\alpha}$ such that the roots of the equation

$$\sinh^2(\tau\omega) = \tau^2 \sin^2 \omega$$

see [6] (7,2,2,1), except for $-i$ and 0 all have sufficiently small (negative, large absolute value) imaginary value. Note $1 + i\tau = -i\lambda$.

Remark 4: For the H_0^2 interpolation to hold, it now suffices to quote [6], Thm. 7.2.2.3. The H_0^2 interpolation result is thus essentially – along with the reasoning of the proof of the Prop. and a localization of the poles of $R(\lambda)$ for the biharmonic – a consequence of the analysis in Grisvard's monograph [6]. Such an analysis was first done in [8] and [12] for Stokes problem (which via the stream function connects to the biharmonic problem).

For $m > 2$ we shall employ a recent regularity theorem in [5].

Lemma 2 *Assume that $k \ge -m$, $k \notin -m + \{\frac{1}{2}, \frac{3}{2}, \cdots, m - \frac{1}{2}\}$ and let $f \in H^k(\tilde{\Omega})$ ($\tilde{\Omega}$ is defined above). Then, for sufficiently small $\tilde{\alpha}$, the solution $u \in H_0^m(\tilde{\Omega})$ to*

$$(-\Delta)^m u + u = f \text{ in } \tilde{\Omega} \tag{3.3}$$

belongs to $H^{k+2m}(\tilde{\Omega})$ and $\|u\|_{H^{k+2m}} \le C\|f\|_{H^k}$.

Proof: The statement is really a corollary of Lemma 1 and a recent shift theorem in [5], Corollary (5.2) proven in [5] §10. It is stated for a cone in Thm. (1.11). In order to apply this result we select the angle $\tilde{\alpha}$ sufficiently small that Lemma 1 ensures that $R(\lambda)$ has no poles in the strip $\mathrm{Im}\lambda \in [m-1, t-2]$. This in turn implies Dauge's condition (C2*) ($\sim$ (R2)) as follows: If $-i\lambda \in \mathbf{N}$, then [5] Cor. (4.6') yields (C2*) and if $-i\lambda \notin \mathbf{N}$, then Cor. (4.9) along with Cor. (4.15) and the fact that $a = 2$ (see [5] p. 39) concludes the proof.

////

Remark 5: For $m > 2$ it is possible to prove Lemma 2 directly from [10] when $k \geq -1$, $k \in \mathbf{Z}$. [10] has the result for $f \in H_0^k$ and $k \in \mathbf{N}$. Then it is possible to prove for $f \in H^k$ when $k \geq -1$ by generalizing the trace theorem 7.2.2.3 in [6] for a domain $\tilde{\Omega}$ with one corner C of sufficiently small angle $\tilde{\alpha}$ between the two linear pieces Γ_j, $j = 1, 2$. First, by this, one finds $v \in H^{k+2m}(\tilde{\Omega}) \cap H_0^m(\tilde{\Omega})$ such that

$$(-\triangle)^m v + v - f \in H_0^k(\tilde{\Omega}) \tag{3.4}$$

Then one applies to $w = u - v$, Kondrat'evs result (when $f \in H_0^k$) with Lemma 1 choosing $\tilde{\alpha}$ sufficiently small that $w \in H^{k+2m}$ where $v \in H^{k+2m}$ is the solution to (3.4). In [6] the generalization of Kondrat'ev's weighted spaces is given for $k = -1$. It seems difficult however to go to all the remaining negative integers.

4. Acknowledgements

The author wishes to expressly thank a number of people at University of Maryland, College Park for some very helpful discussions: Michael Vogelius, Ivo Babuška, Bruce Kellogg, John Osborn, and Tobias von Petersdorff. Also thanks to I.P.S.T., College Park, for giving me the opportunity for such discussions.

References

[1] I. Babuška and M. Dorr. Error estimates for the combined h and p versions of the finite element method. *Numer. Math.*, 37:257–277, 1981.

[2] I. Babuška, S. Jensen, and M. Suri. On locking in the performance of the p-version of the finite element method for Stokes problem. in prep., 1989.

[3] I. Babuška, R.B. Kellogg, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, 33:447–471, 1979.

[4] J. Bergh and J. Löfstrom. *Interpolation Spaces*. Springer, 1976.

[5] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains*, volume 1341 of *Lecture Notes in Math.* Springer, 1988.

[6] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, 1985.

[7] I.N. Katz and D.W. Wang. The p-version of the finite element method for problems requiring C^1-continuity. *SIAM J Num Anal*, 22:1082–1106, 1985.

[8] R.B. Kellogg and J.E. Osborn. A regularity result for the Stokes problem in a convex polygon. *J. Functional Anal.*, 21:397–431, 1976.

[9] V. A. Kondrat'ev. Boundary problems for parabolic equations in closed domains. *Trans. Mosc. Math. Soc.*, 15:450–504, 1966.

[10] V. A. Kondrat'ev. Boundary problems for elliptic equations in domains with conical or angular points. *Trans. Mosc. Math. Soc.*, 16:227–313, 1967.

[11] Y. Maday. Analysis of spectral projections in multi-dimensional domains. Private communication, May 1989.

[12] J.E. Osborn. Regularity of solutions of the Stokes problem in a polygonal domain. In B. Hubbard, editor, *Numerical Solution of Partial Differential Equations - III*, pages 393–411. Academic Press, 1976.

[13] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*, volume 30 of *Princeton Mathematical Series*. Princeton University Press, 1970.

[14] M. Suri. The p-version of the finite element method for elliptic equations of order $2l$. Technical report, University of Maryland Baltimore County, 1987. To appear in MMAN, RAIRO.

[15] M. Vogelius and G. Papanicolaou. A projection method applied to diffusion in a perodic structure. *SIAM J Appl Math*, 42:1302–1322, 1982.